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ADDENDUM

Inverse bremsstrahlung absorption in large radiation fields during binary collisions—classical theory II(*b*). Summed rate coefficients for Coulomb collisions

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Abstract. The summation of the expressions obtained previously for the value of the absorption coefficient is performed analytically, and their asymptotic expansion derived. These results are shown to be in agreement with similar expressions obtained by other methods. It is concluded that this form represents a generalization of the Kramers–Gaunt low-field absorption coefficient.

In a recent paper (Pert 1976) we derived an exact expression for the electron absorption coefficient of radiation in a fully ionized plasma at arbitrary intensity using the classical approximation. In this addendum we show how the expression obtained may be simplified to give a general form of the Kramers–Gaunt formula (Kramers 1923, Gaunt 1930). In the earlier work we found that the absorption coefficient κ was given by

$$\kappa = \kappa_0(\mathscr{S}_1 \ln \Delta + \eta \mathscr{S}_2)$$

where the sums $\mathscr{S}_1$ and $\mathscr{S}_2$ and the terms κ_0, Δ and η have been given earlier.

The series $\mathscr{S}_1$ may be evaluated by the use of Kummer's transformation on the confluent hypogeometric function to give:

$$\tfrac{1}{3}x\mathscr{S}_1 = \sum_{n=1}^{\infty} \frac{1}{n!(2n+1)x^n} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}+n)_m(-x)^m}{m!(n+1)_m}$$

which may be inverted to yield:

$$\tfrac{1}{3}x\mathscr{S}_1 = \sum_{m=1}^{\infty} \frac{(-x)^m\Gamma(m+\frac{1}{2})}{(m!)^2\Gamma(\frac{1}{2})} \sum_{n=1}^{m} \frac{(-m)_n}{(\frac{3}{2})_n}.$$

But

$$\sum_{n=0}^{m} (-m)_n/(\tfrac{3}{2})_n = {}_2F_1(1, -m; \tfrac{3}{2}; 1) = \tfrac{1}{2}/(\tfrac{1}{2}+m).$$

Hence we obtain:

$$\mathscr{S}_1 = {}_2F_2(\tfrac{3}{2}, \tfrac{3}{2}; \tfrac{5}{2}, 2; -x)$$

which is identical to the function F obtained by Brysk (1975).

To calculate $\mathscr{S}_2$ we proceed in a similar manner, using Kummer's transformation and replacing the digamma function by the appropriate sum to yield:

$$\mathscr{S}_2 = -\gamma\mathscr{S}_1 + \frac{3}{x}\sum_{m=2}^{\infty}\frac{(-x)^m\Gamma(m+\frac{1}{2})}{(m!)^2\Gamma(\frac{1}{2})}\sum_{n=2}^{m}\frac{(-m)_n}{(\frac{3}{2})_n}\sum_{l=2}^{n}\frac{1}{l-1}.$$

Inverting the last two sums we obtain:

$$\sum_{n=2}^{m}\frac{(-m)_n}{(\frac{3}{2})_n}\sum_{l=2}^{n}\frac{1}{l-1} = \frac{-m}{\frac{1}{2}+m}\sum_{l=1}^{m-1}\frac{(-m+1)_l(l-1)!}{l!(\frac{3}{2})_l}.$$

Consider the sum on the right-hand side. In the previous case we could evaluate this term by equating it to a hypogeometric function of argument 1, but in this case one of the orders is zero. We therefore write

$$\sum_{l=1}^{m}\frac{(-m)_l(l-1)!}{l!(\frac{3}{2})_l} = \lim_{\Delta\to 0}\frac{1}{\Delta}({}_2F_1(\Delta, -m; \tfrac{3}{2}; 1) - 1) = \psi(\tfrac{3}{2}) - c(\tfrac{3}{2}+m)$$

to obtain:

$$\mathscr{S}_2 + \gamma\mathscr{S}_1 = -\frac{\mathrm{d}}{\mathrm{d}\Delta}\,{}_2F_2(\tfrac{3}{2}, (\tfrac{3}{2}+\Delta); \tfrac{5}{2}, 2; -x)|_{\Delta=0}.$$

The true asymptotic expansions of $\mathscr{S}_1$ and $\mathscr{S}_2$ may now be obtained either by the Mellin–Barnes integral representation as suggested by Jorna (1975) or by the transformation used by Brysk (1975); namely,

$$\mathscr{S}_1 \to \tfrac{3}{2}x^{-3/2}\pi^{-1/2}\Big(\ln x + \gamma + 4\ln 2 - 2\pi^{-1}\sum\Gamma(\tfrac{3}{2}+n)\Gamma(\tfrac{1}{2}+n)x^{-n}/n!n\Big)$$

$$\mathscr{S}_2 \to \tfrac{3}{2}x^{-3/2}\pi^{-1/2}\Big(\tfrac{1}{2}(\ln x + 2\ln 2)^2 - (\gamma + \ln 2)^2/2 - 2(\gamma - 1)(\gamma + 2\ln 2 - 2)$$
$$-\tfrac{1}{12}\pi^2 + 2\pi^{-1}\sum\Gamma(\tfrac{3}{2}+n)\Gamma(\tfrac{1}{2}+n)x^{-n}/n!n[\gamma - \psi(\tfrac{3}{2}) + \psi(\tfrac{3}{2}+n) + \psi(\tfrac{1}{2}+n)$$
$$-\ln x - 1/n]\Big).$$

We remark that these results are in good agreement with the numerical results published in our earlier paper. Indeed comparison checks show that even the approximate formulae for large values of x are accurate to better than a few per cent. Comparing the analytic forms of the asymptotic expansion with those proposed earlier, we see that the previous suggestions are indeed correct.

It was noted that our previous work was valid provided the appropriate electron collision cross section σ could be written in the form:

$$\sigma = \sigma_0 \ln(\alpha v^\eta)/v^4$$

although the calculations were specifically performed for classical electron collisions where:

$$\frac{Ze^2}{\hbar v} \gg 1 \qquad \sigma_0 = \frac{4\pi e^4 Z^2}{m^2 v^4} \qquad \alpha = \frac{Ze^2}{m\omega} \qquad \eta = 3.$$

However, in a separate paper (Pert 1975) we have shown that extreme quantal electrons under the conditions of the Born approximation may also be treated in this way with:

$$\frac{Ze^2}{\hbar v} \ll 1 \qquad \sigma_0 = \frac{4\pi e^4 Z^2}{m^2 v^4} \qquad \alpha = \frac{2m}{\hbar\omega} \qquad \eta = 2.$$

Indeed we may note that since:

$$\lim_{y\to 0} \frac{1}{y}\kappa_0(y) \sinh y \to \gamma - \ln(\hbar\omega/4kT)$$

the classical approximation reproduces the results of Brysk (1975) and Osborn (1972) in the Born approximation limit, apart from the higher-order sum in $\mathscr{S}_2$. The classical approximation allows the generalization of this result for both slow and fast electrons.

Indeed, these results may be expressed more generally in terms of the Gaunt factor. It is well known (Oster 1961) that the exact low-field calculation (Kramers–Gaunt formula) leads to the introduction of a Gaunt factor which in its classical† and Born limits is almost identical to the logarithmic terms obtained here. Thus the logarithmic term in equation (1) is identified as the averaged Gaunt factor. (We note that in the classical case there is a difference between the Gaunt factor and ln Δ:

$$\bar{g} - \tfrac{1}{2}\ln\Delta = \ln 2 - \gamma = 0{\cdot}115931$$

which is an error associated with the sharp outer cut-off (Pert 1972): a more accurate calculation is needed to resolve this issue, but for the present we believe it is probably better to use $\bar{g}$ at all field strengths rather than $\frac{1}{2}\ln\Delta$‡.) We therefore generalize equation (1) as:

$$\kappa = 2\kappa_0(\bar{g}\mathscr{S}_1 + \bar{\eta}\mathscr{S}_2)$$

which we believe is the generalized form of the Kramers–Gaunt formula (Kramers 1923, Gaunt 1930) where the Gaunt factor $\bar{g}$ and the factor $\bar{\eta}$ have the limits:

$$\bar{g} \to \bar{g}_{\text{class}}\ (\text{or } \tfrac{1}{2}\ln\Delta) \qquad \bar{\eta}\to\tfrac{3}{2} \qquad \frac{Z^2e^4m}{\hbar^2kT_e} \gg 1$$

$$\bar{g} \to \bar{g}_{\text{Born}} \qquad \bar{\eta}\to 1 \qquad \frac{Z^2e^4m}{\hbar^2kT_e} \ll 1$$

where the values of the average Gaunt factor in the classical and Born limits are given by Oster (1961, equations (162) and (163)). We may infer that at intermediate values of $Z^2e^4m/\hbar^2kT_e$ little error is incurred by using the low-field value of the Gaunt factor (Karzas Latter 1961) and a suitable estimate of $\bar{\eta}$.

In a recent letter Geltman (1975) has derived a correction factor associated with the infinite range of the $1/r$ potential. However, he has pointed out that if the potential has a finite range, this correction to the Born approximation does not appear. In reality the Coulomb potential is always cut-off at some finite range—the Debye shielding length or the interparticle distance—so that the correction is not applicable in practice and the Born approximation result of Brysk (1975) can be expected to hold. Indeed the applicability of this correction would seem to be limited to cases where the duration of the laser pulse is sufficiently short, that the electron remains within the Coulomb field of a single ion throughout the laser pulse, i.e. that the electron collision time is longer than the pulse length.

† Note that in this context, classical refers to the electron collisional behaviour, not to the classical model of absorption.

‡ This same difference appears between the exact binary collision expression (Oster 1961) and the Vlasov equation calculation (Dawson and Oberman 1962) where the outer cut-off is treated properly but a sharp inner one is used.

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